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FINITE ELEMENT EMBEDDING WITH OPTICAL INTERFERENCE

D. K. EDWARDS*
School of Engineering, UCLA, Los Angeles, CA 90024, U.S.A.

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Abstract—A convenient algorithm for calculating the overall reflectivity from, and the absorptivity in each layer of, a multi-thin-film array is reported. To account properly for interference the radiative transfer problem is formulated in terms of the amplitude of the electric field vector rather than the radiant flux or intensity. Commencing with a substrate, the system coefficients are constructed by finite element embedding. When the embedding builds to the top of the array, the field amplitude is assigned a normalized value of unity, and the remaining amplitudes are back calculated step by step. The Poynting Theorem permits the normalized fluxes to be found, and differencing then gives the absorptivity of each layer. Absorbing media, polarization, interference, and oblique incidence are all fully included. The absorptivity-per-layer values are needed in analyzing the cooling requirements of laser mirrors and absorbers used with high concentration solar collectors.

NOMENCLATURE

A_n^+,	amplitude of electric field of upgoing wave above the nth interface;
A_n^-,	amplitude of electric field of downcoming wave above the nth interface;
B_n^+,	amplitude of electric field of upgoing wave below the nth interface;
B_n^-,	amplitude of electric field of downcoming wave below the nth interface;
b_n,	z wavenumber for medium n;
d_n,	thickness of nth medium;
E,	electric field strength;
H,	magnetic field strength;
k,	z-direction unit vector;
k_n,	absorptive index of medium n;
k_t,	thermal conductivity;
i,	unit imaginary number;
L_r,	radiation mean free path;
m_n,	medium attenuation and phase change parameter, equation (6);
N,	total number of interfaces (and films);
n_n,	refractive index of medium n;
q^+,	radiosity [W/m^2];
q^-,	irradiation [W/m^2];
q,	net flux [W/m^2];
R_n,	reflection coefficient, equation (11);
r_n,	Fresnel reflection coefficient for top of nth interface;
r'_n,	Fresnel reflection coefficient for bottom of nth interface;
$r_{i,j}$,	Fresnel reflection coefficient for interface between media i and j;
t,	time;
t_n,	Fresnel transmission coefficient for top of nth interface;
t'_n,	Fresnel transmission coefficient for bottom of nth interface;
T,	temperature;
T_n,	transmission coefficient, equation (10);
z,	distance normal to surface.

Greek symbols

α,	absorptivity;
α_t,	thermal diffusivity;
δ,	wall thickness;
ε,	emissivity;
θ,	angle from normal;
λ,	wavelength in vacuum;
σ,	Stefan–Boltzmann constant.

Subscripts

i, j, m, n,	ith, jth, mth or nth interface or medium;
o,	incident radiation;
(p),	polarization parallel to plane of incidence;
(s),	polarization perpendicular to plane of incidence;
t,	thermal;
x, y, z,	vector components.

Superscripts

+,	upgoing;
−,	downcoming radiation;
*,	complex conjugate;
˜,	complex quantity (for emphasis).

* Present address: School of Engineering, UCI, Irvine, CA 92717, U.S.A.

INTRODUCTION

CONSIDER the problem of predicting the transient or steady temperature in a solid exposed on one face to a high flux of radiant energy. If the time of exposure t is long enough so that the depth of a thermal wave $\sqrt{(\alpha_t t)}$, where α_t is the thermal diffusivity, is large compared to the radiant exponential decay length L_r in the solid, and the thickness of the solid δ is also large compared to L_r, one treats the problem as a standard conduction problem subject to a radiation boundary condition

$$-k_t \frac{\partial T}{\partial z}\bigg|_0 = \alpha q^- - \varepsilon\sigma T^4$$

where α is the surface absorptivity, q^- is the irradiation (W/m^2) and the other terms have their usual meaning as shown in the Nomenclature. However, when the exposure time is short, the solid thin, and/or the solid material is not highly opaque, the distance L_r over which the radiation is attenuated may not be small compared to $\sqrt{(\alpha_t t)}$ or δ, and the level of analysis above is inadequate. Instead, it is necessary to have knowledge of how much radiation is absorbed as a function of depth from the surface and use this knowledge to formulate the problem as conduction with a depth-dependent volume heat source.

The problem becomes more complex when the solid is multilayered, because a simple exponential radiation decay law no longer holds. But multilayer thin films are the rule rather than the exception for high performance laser mirrors and are used on solar collectors, spacecraft thermal control surfaces, and solar cell covers. Thus the problem arises of how to compute the absorbed flux in each layer of a multilayered thin film system.

A similar problem arises in computing the absorbed flux in each layer of a multiglazed solar collector. This problem was addressed by Wijeysundera [1], who formulated it in simultaneous linear equations to be solved by matrix inversion. Siegel [2] had previously used the net radiation (radiosity − irradiation) approach to compute transmission through a stack of plates. A more convenient computational algorithm was found to be finite element embedding [3], an extension [4] of the differential embedding technique developed by Bellman *et al.* (see, e.g. [5]).

In order to apply such a technique to multilayered thin films, it is necessary to account correctly for the fact that the layers have thicknesses that are fixed and comparable to the wavelength of the irradiation. Reflected electromagnetic waves from different elements of the thin film array interfere, constructively or destructively, and that interference is used by the optical designer to obtain the desired result, a very high reflectivity in the case of a laser mirror, or a very high absorptivity in the case of a solar absorber.

The optics of multi-thin-films are well developed. The problem of predicting the overall reflectivity or a multilayer stack or designing a stack to obtain a desired reflectivity has received much attention, e.g. [6]. The problem of predicting the peak absorption in a weakly absorbing dielectric layer has also attracted attention, because peak absorption has been associated with laser-induced damage. An excellent literature survey is given by Bennett and Burge [7]. As stated by them: "When the optical properties of a single- or multilayer-film-coated surface are calculated as a boundary value problem, immensely complicated expressions result." To avoid this complexity Apfel [8] employed a recursion relation, in effect the embedding algorithm that itself can be regarded as a form of the Gauss elimination algorithm. Apfel suggested using the recursion relation to find the immediate boundary conditions on a layer and compute the peak time-averaged field squared. The calculational method is outlined; complete details are not given.

A significant factor in multilayer films identified by Temple *et al.* [9] is absorption in thin (*ca.* 0.1 μm) interfacial layers. The presence of an interfacial layer was inferred from two types of measurements. In one, a film showed an absorptance virtually independent of thickness at a value higher than the bare substrate. In another, a wedge-shaped film was shown to display a residual jump in absorptance when measurements were extrapolated to zero thickness. Bennett and Burge [7] identified scattering from imperfections as another factor increasing absorption.

In what follows the problem of finding the absorptivity in each layer of a multi-thin-film array is solved by applying the finite element embedding algorithm. The algorithm is developed and presented in full detail, and sample calculations are shown. Arbitrarily absorbing media, polarization, and oblique incidence are all fully included. Interfacial layers can be included to the extent that their properties are known, but no provision has been made for scattering.

FUNDAMENTAL RELATIONS

As can be seen in Fig. 1, a layered stack of thin films consists of a number of materials of various thicknesses. These materials are media for transmission of the electromagnetic radiation and are numbered from 1 to N from the substrate up to the top of the stack. A final medium, air or space, exists above the stack and is given medium number $N + 1$. Above each medium is an interface between it and the next one. These interfaces are given an index number as shown in the figure. The numbering from the bottom to the top facilitates using the embedding algorithm.

Reflection and transmission occurs at each interface, and absorption occurs upon the passage of radiation through each layer. Because the layers have regular thicknesses, precise to a small fraction of a wavelength (in contrast to say a piece of window glass), the electromagnetic waves interfere: for example the wave reflecting from the top of the ith interface and one reflected from the $i - 1$ interface below and transmitted through the ith interface have different phases because of the longer upward and downward path taken by the latter. To account for the regularity in the phase differences between the waves it is necessary to formulate in terms of electric or magnetic amplitude instead of radiant flux as is usually done by the radiation heat transfer worker. Nevertheless, the radiosity − irradiation approach usually favored by the heat transfer worker can be retained with considerable advantage to the ray tracing approach sometimes taken by the optical worker.

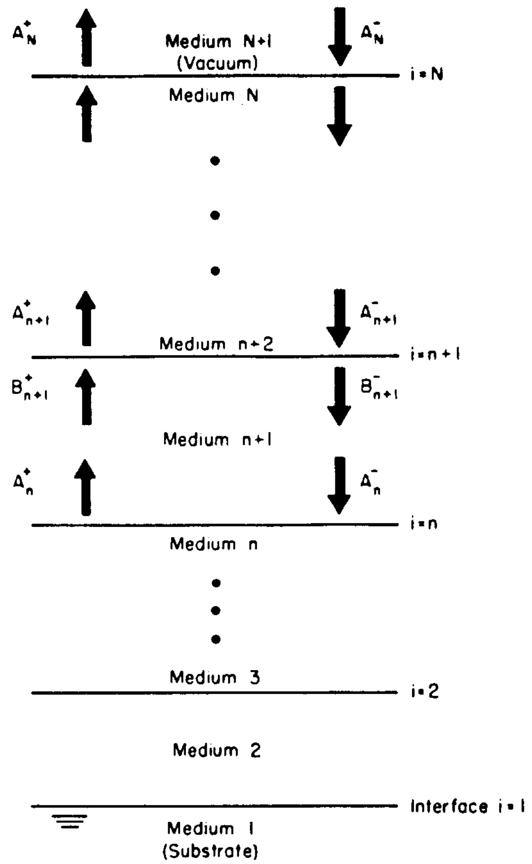

FIG. 1. Multifilm interface and medium number assignments.

Taking an irradiation radiosity approach, one distinguishes between the total upward traveling wave (the radiosity) and the total downward traveling wave (the irradiation). Above the nth interface the electric field amplitude of the upward wave is denoted A_n^+ as shown in Fig. 1, and the downward wave there is A_n^-. The upward wave traverses medium $n+1$ and thus experiences some attenuation (absorption) and phase shift before arriving beneath the $n+1$ interface. Beneath the $n+1$ interface the upward wave electric field amplitude is denoted B_{n+1}^+. Similarly the downward-going wave beneath that interface is denoted B_{n+1}^- as shown in the figure.

To summarize, the subscript of an interfacial quantity denotes the interface, and the subscript of a material property or layer thickness denotes the medium. The ith interface lies above the ith medium.

The Fresnel coefficients [10, 11] are interfacial quantities that relate the amplitudes above and below each interface. Denote as r_n the reflection coefficient for the downward wave above the n interface and the transmission coefficient as t_n. Let those for the upward wave below the n interface be r'_n and t'_n. The single interface subscript is used in place of the usual double medium subscript [11] as follows:

$$r_n = r_{n+1,n}, \quad t_n = t_{n+1,n}$$
$$r'_n = r_{n,n+1}, \quad t'_n = t_{n,n-1}, \qquad (1a\text{–}d)$$

where for p-polarization

$$r_{i,j(p)} = \frac{\tilde{n}_j \cos\tilde{\theta}_i - \tilde{n}_i \cos\tilde{\theta}_j}{\tilde{n}_j \cos\tilde{\theta}_i + \tilde{n}_i \cos\tilde{\theta}_j}, \qquad (2a)$$

$$t_{i,j(p)} = \frac{2\tilde{n}_i \cos\tilde{\theta}_i}{\tilde{n}_j \cos\tilde{\theta}_i + \tilde{n}_i \cos\tilde{\theta}_j} \qquad (2b)$$

and for s-polarization

$$r_{i,j(s)} = \frac{\tilde{n}_i \cos\tilde{\theta}_i - \tilde{n}_j \cos\tilde{\theta}_j}{\tilde{n}_i \cos\tilde{\theta}_i + \tilde{n}_j \cos\tilde{\theta}_j}, \qquad (2c)$$

$$t_{i,j(s)} = \frac{2\tilde{n}_i \cos\tilde{\theta}_i}{\tilde{n}_i \cos\tilde{\theta}_i + \tilde{n}_j \cos\tilde{\theta}_j}. \qquad (2d)$$

The tilde denotes complex arithmetic. Quantities A_n, B_n, E, H, m_n, r_n, R_n, t_n and T_n are also complex. Snell's law relates the complex sin $\tilde{\theta}_i$ to the real angle of incidence θ_0.

$$\tilde{n}_i \sin\tilde{\theta}_i = \tilde{n}_{N+1} \sin\tilde{\theta}_{N+1} = \sin\theta_0, \qquad (3a)$$

$$\cos\tilde{\theta}_i = \sqrt{1 - \sin^2\tilde{\theta}_i} = \sqrt{1 - (\sin^2\theta_0/\tilde{n}_i^2)}. \qquad (3b)$$

The complex index of refraction is

$$\tilde{n}_j = n_j - \mathrm{i}k_j \qquad (4)$$

where i is the unit imaginary number $\sqrt{(-1)}$. The (real) refractive index is n_j, and the absorptive index is k_j in the jth medium. With this notation the A_n and B_n amplitudes are related as follows:

$$A_n^+ = r_n A_n^- + t'_n B_n^+, \qquad (5a)$$

$$B_n^- = r'_n B_n^+ + t_n A_n^-. \qquad (5b)$$

Equation (5a) is the mathematical statement of the fact that the upward wave above the nth interface is formed by reflection of the downward wave from above plus transmission of the upward wave from below. Similarly eqn. (5b) states that the downward wave below is formed by reflection from below of the upward wave beneath the interface plus transmission of the downward wave arriving from above.

The complex amplitude change in phase and magnitude upon traversal (upward or downward) of medium $n+1$ is accounted for by multiplier m_{n+1} [11]

$$m_{n+1} = \exp(-\mathrm{i}\tilde{n}_{n+1} \cos\tilde{\theta}_{n+1}\, 2\pi d_{n+1}/\lambda) \qquad (6)$$

where λ is the wavelength in medium $N+1$, assumed to be a dielectric (vacuum or low density gas) with unit refractive index, and d_{n+1} is the thickness of medium $n+1$. Accordingly

$$B_{n+1}^+ = m_{n+1} A_n^+, \qquad (7a)$$

$$A_n^- = m_{n+1} B_{n+1}^-. \qquad (7b)$$

With A_{N+1}^- prescribed for a given component of polarization and $B_1^+ = 0$ there are 4 unknowns (A_n^+, A_n^-, B_n^+, B_n^-) for each of $N-2$ interfaces ($n \neq 1$ or N) and 3 unknowns at interfaces 1 and N, making a total of $4N-2$ unknowns. There are two equations [5(a) and 5(b)] for each interface (n runs from 1 to N) giving $2N$ such equations and two equations [7(a) and 7(b)] for each layer (media 2 to N) giving $2(N-1)$ more. The total of $4N-2$ equations can be solved by matrix inversion, but the embedding algorithm is more convenient.

EMBEDDING

For a given component of polarization, imagine that amplitude A_{n+1}^- is known. Let all amplitudes be normalized to that value. Suppose also that the linear relation between A_n^+ and A_n^- is known; call it R_n, then

$$A_{n+1}^- \equiv 1, \tag{8}$$

$$A_n^+ = R_n A_n^-. \tag{9}$$

The outgoing wave amplitude A_{n+1}^+ normalized by A_{n+1}^- is R_{n+1} and is sought. We have equation (9) and equations 5(a, b) and 7(a, b). To recapitulate:

$$R_{n+1} = r_{n+1} + t'_{n+1} B_{n+1}^+, \tag{5a}$$

$$B_{n+1}^- = r'_{n+1} B_{n+1}^+ + t_{n+1}, \tag{5b}$$

$$B_{n+1}^+ = m_{n+1} A_n^+, \tag{7a}$$

$$A_n^- = m_{n+1} B_n^-, \tag{7b}$$

$$A_n^+ = R_n A_n^-. \tag{9}$$

Substitute equation (7a) into (5b) and the result into equation (7b). Introduce equation (9) and solve for the normalized A_n^-:

$$A_n^- \equiv T_{N+1} = \frac{m_{n+1} t_{n+1}}{1 - m_{n+1}^2 r'_{n+1} R_n}. \tag{10}$$

Substitute equation (10) into equation (9) and the result into equation (5a) to find the desired recursion formula

$$R_{n+1} = r_{n+1} + \frac{m_{n+1}^2 t_{n+1} t'_{n+1} R_n}{1 - m_{n+1}^2 r'_{n+1} R_n}. \tag{11}$$

The embedding procedure commences with $R_1 = r_1$ and $T_1 = t_1$. Then equations (10) and (11) generate T_2 and R_2 and then T_3 and R_3, and so on, up to T_N and R_N. These (complex) coefficients are computer calculated and stored for each component of polarization.

PEELING

Once the set of R_n and T_n are found, the wave amplitudes normalized to $A_N^- = 1$ can be back calculated by peeling off one interface after another. From equation (10) written for $n = m - 1$,

$$A_{m-1}^- = T_m A_m^-. \tag{10}$$

Commencing with $m = N$ and peeling back to $N-1$, $N-2, \ldots, 1$, one finds the set of A_n.

When the amplitudes are found (or as soon as each one is found) the net flux can be found above or below the corresponding interface. Poynting's Theorem states that the normalized flux is

$$q = \mathrm{Re}\{k \cdot \tilde{E} \times \tilde{H}^*\}/q_o \tag{12}$$

where $\tilde{E}$ is the complex electric field vector, $\tilde{H}^*$ is the complex conjugate of the magnetic field amplitude vector, k is the unit vector in the z-direction normal to the interface, and q_o is the incident flux normalizing factor:

$$q_o = (A_{n+1}^-)^2 \cos\theta_{N+1} = \cos\theta_o. \tag{13}$$

The z component of the vector cross product $\tilde{E} \times \tilde{H}^*$ is composed of two terms, $E_x H_y^*$ and $-E_y H_x^*$, the former being the p-polarization contribution, and the latter the s-polarization component:

$$q_{n(p)} = \mathrm{Re}\{E_x H_y^*\}_n / \cos\theta_o, \tag{14a}$$

$$q_{n(s)} = \mathrm{Re}\{-E_y H_x^*\}_n / \cos\theta_o. \tag{14b}$$

Above the nth interface in medium $n+1$ the complex z-direction-cosine is $\cos\tilde{\theta}_{n+1}$, and the complex z-wavenumber is $\tilde{b}_{n+1} = \tilde{n}_{n+1} \cos\tilde{\theta}_{n+1}$. Maxwell's equations show that

$$E_x = \cos\tilde{\theta}_{n+1}[A_{n(p)}^- - A_{n(p)}^+], \tag{15a}$$

$$E_y = A_{n(s)}^- + A_{n(s)}^+, \tag{15b}$$

$$H_x = \tilde{b}_{n+1}[A_{n(s)}^+ - A_{n(s)}^-], \tag{15c}$$

$$H_y = \tilde{n}_{n+1}[A_{n(p)}^- + A_{n(p)}^+]. \tag{15d}$$

As a check on the accuracy and one guard against programming error, it is as well to compute $q_{n(p)}$ and $q_{n(s)}$ below an interface and compare them to the values above. Since interfacial absorption is provided for only by having discrete finite (albeit thin) layers, the values above and below the interface must agree to within round-off error. Below the interface

$$E_x = \cos\tilde{\theta}_n[B_{p,n}^- - B_{p,n}^+], \tag{16a}$$

$$E_y = B_{n(s)}^- + B_{n(s)}^+, \tag{16b}$$

$$H_x = \tilde{b}_n[B_{n(s)}^+ - B_{n(s)}^-], \tag{16c}$$

$$H_y = \cos\tilde{\theta}_n[B_{n(p)}^+ + B_{n(p)}^-]. \tag{16d}$$

Equations 14(a) and (b) give the fluxes.

The partial absorptivity of each layer can then be found by differencing. For the nth medium ($n = 2, N$)

$$\alpha_{n(p)} = q_{n(p)} - q_{n-1(p)}, \tag{17a}$$

$$\alpha_{n(s)} = q_{n(s)} - q_{n-1(s)}. \tag{17b}$$

For an opaque substrate $n = 1$ the absorptivity is just the normalized flux entering it. For irradiation that is (on the average) unpolarized the partial absorptivity of the nth layer is

$$\alpha_n = \tfrac{1}{2}[\alpha_{n(p)} + \alpha_{n(s)}]. \tag{18}$$

The system absorptivity is, of course,

$$\alpha_{1-N} = \sum_{n=1}^{N} \alpha_n = 1 - R_N. \tag{19}$$

Equation (19) (which can be written for each component of polarization as well as for the average) really consists of two relations, serving as yet another check upon the calculation.

Table 1. Sample results for a five-pair high-low half-wave multilayer system

Interface or medium n	Normalized flux above and below interface $q_{n(p)}$	$q_{n(s)}$	Net flux for unpolarized irradiation above and below interface q_n	Partial absorptivity of medium α_n
1	0.914408	0.873523	0.893966	0.893966
2	0.918523	0.877468	0.897996	0.004030
3	0.922745	0.881573	0.902159	0.004163
4	0.926900	0.885556	0.906228	0.004069
5	0.931147	0.889683	0.910415	0.004187
6	0.935354	0.893723	0.914538	0.004123
7	0.939613	0.897852	0.918733	0.004195
8	0.943881	0.901959	0.922920	0.004187
9	0.948147	0.906085	0.927116	0.004196
10	0.952475	0.910259	0.931367	0.004251
11	0.956754	0.914388	0.935571	0.004204

Substrate properties: $n_1 = 1.5$, $k_1 = 0.0$.
Even ($n = 2, 4, 6, 8, 10$) medium properties: $n_2 = 1.4$, $k_2 = 0.00093$, $d_2 = 3.75\,\mu m$.
Odd ($n = 3, 5, 7, 9, 11$) medium properties: $n_3 = 2.1$, $k_3 = 0.00143$, $d_3 = 2.50\,\mu m$.
Wavelength: $\lambda = 10.5\,\mu m$; angle of incidence: $\theta_o = 30°$.

SAMPLE RESULTS

Optical coatings for laser mirrors, solar absorbers, solar cell cover-glasses, and spacecraft thermal control surfaces often consist of arrays of pairs of materials. At times thin layers of a third material are introduced to provide a diffusion barrier or to improve adhesion. In the preceding development any number of different materials may be accommodated when their thicknesses, optical properties, and order of stacking are known. Two examples are presented, one for a half-wave stack of pairs of materials, and the other for a single pair.

Table 1 shows results for an array of five pairs of one-half wave high-n-low-n layers. The one-half wave layer has thickness such that $n_n d_n/\lambda$ in equation (6) is $\frac{1}{2}$. It minimizes reflection and thus transmits a high fraction of the incident irradiation into the substrate. In this example the absorption per layer is small. A normalized flux of 0.935571 is transmitted through the top interface ($n = 11$), $1 - 0.935571 = 0.064429$ is reflected, and 0.931367 is transmitted through the second interface ($n = 10$) etc. The difference, $0.935571 - 0.931367 = 0.004204$, is absorbed in the top layer, medium 11, and so on. The bulk of the irradiation, a fraction equal to 0.893966, passes into the substrate. To (at least) the number of digits shown,

Table 2. Sample results for a quarter-wave absorber/emitter

Wavelength λ (μm)	Partial absorptivities of media α_1	α_2	α_3	Absorptivity of system $\alpha_1 + \alpha_2 + \alpha_3$
3.00	0.1656	0	0.1295	0.2951
3.25	0.0744	0	0.4567	0.5311
3.50	0.0430	0	0.5716	0.6146
3.75	0.0336	0	0.0604	0.6400
4.00	0.0317	0	0.6122	0.6439
4.50	0.0406	0	0.5737	0.6143
5.00	0.0697	0	0.4488	0.5185
5.50	0.1282	0	0.1904	0.3186
6.00	0.1564	0	0.0504	0.2068
7.00	0.0720	0	0.4046	0.4766
8.00	0.0386	0	0.5509	0.5895
9.00	0.0273	0	0.6003	0.6277
10.00	0.0249	0	0.6157	0.6406
12.00	0.0191	0	0.6326	0.6517
15.00	0.0186	0	0.6274	0.6460
18.00	0.0198	0	0.6125	0.6323
24.00	0.0237	0	0.5728	0.5965
30.00	0.0277	0	0.5279	0.5557

Substrate properties: $n_1 = k_1 = 10(\lambda/\lambda_o)^{1/2}$, $\lambda_o = 12\,\mu m$.
Medium 2 properties: $n_2 = 1.5$, $k_2 = 0$, $d_2 = 2.0\,\mu m$.
Medium 3 properties: $n_3 = k_3 = 10(\lambda/\lambda_o)^{1/2}$, $d_3 = 0.036\,\mu m$.
Angle of incidence: $\theta_o = 30°$ unpolarized irradiation.

the below-the-interface fluxes agree with the above-the-interface ones.

Table 2 shows a quarter-wave emitter used to enhance the emission from the back side of a solar sail [12] or sunshield. The solar sail is to propel a space vehicle by solar–photon recoil. Recall that the photon has energy hc/λ and momentum h/λ [13]. Thus on the solar side of the sail, a high reflectivity metal must be used. However, such a metal will inevitably have also a high solar absorptivity-to-emissivity ratio and may get too hot in the space environment. However, if the required structural strength of the sail is achieved by using a dielectric backing instead of using a metal foil, then the back-side emissivity can be increased. A low-weight system can be made (as viewed from the back) of a metal substrate, a quarter-wave dielectric spacer ($n_2 d_2/\lambda_o = \frac{1}{4}$), and a thin absorbing–emitting top layer. Of course, while the system maximizes the emissivity at wavelength λ_o it minimizes it at $\lambda_o/2$. The total hemispherical emissivity is the average over the Planck spectrum for the service temperature.

Table 2 shows the behavior expected. At $\lambda_o = 12\,\mu$m the system absorptivity–emissivity is a local maximum of 0.6517, of which 0.6236 is contributed by the thin outer layer and only 0.0191 by the substrate. At wavelengths longer than λ_o the partial absorptivity of the outer layer falls while that of the substrate rises somewhat. At $6\,\mu$m and again at $3\,\mu$m the system absorptivity is a minimum with the bulk of the residual absorptivity being contributed by the substrate. Because Hagen–Rubens optical behavior was assumed ($n = k = C\lambda^{1/2}$), the minimum at $3\,\mu$m is greater than that at $6\,\mu$m.

SUMMARY

The concept of finite element embedding previously used to treat layers thick (thus irregular) compared to wavelength has been shown to be applicable to layers thin and regular compared to wavelength, provided amplitudes are used in the two-stream equations instead of radiant fluxes. Complete details of the derivation and sample calculations presented here make it possible for the thermal analyst to readily calculate the partial absorptivity per layer.

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ASSEMBLAGE D'ELEMENTS FINIS AVEC INTERFERENCE OPTIQUE

Résumé—On reporte un algorithme convenable pour le calcul de la réflectivité globale et l'absorptivité de chaque couche d'un arrangement de fines couches multiples. Pour tenir compte de l'interférence, le problème de transfert radiatif est formulé en fonction de l'amplitude du vecteur champ électrique plutôt que le flux radiatif ou l'intensité. Les coefficients du système sont construits par association d'éléments finis en commençant par un substrat. Quand l'association s'élève au sommet de l'arrangement, l'amplitude du champ a une valeur normalisée unitaire et les autres amplitudes sont recalculées pas à pas. Le théorème de Poynting permet de trouver les flux normalisés et en les différenciant on obtient l'absorptivité de chaque couche. Les milieux absorbants, la polarisation, l'interférence, l'incidence oblique sont tous inclus. Les valeurs d'absorptivité par couche sont nécessaires dans l'analyse des besoins de refroidissement des miroirs laser et des absorbeurs utilisés dans les collecteurs solaires à forte concentration.

FINITE ELEMENTE-ANWENDUNG BEI OPTISCHER INTERFERENZ

Zusammenfassung—Es wird ein bequemer Algorithmus zur Berechnung des Gesamtreflexionsvermögens und des Absorptionsvermögens in jeder Schicht einer mehrfachen Anordnung dünner Schichten vorgestellt. Um die Interferenz richtig anzusetzen, wird das Strahlungsaustauschproblem zweckmäßiger in Amplitudentermen des elektrischen Feldvektors dargestellt als in Termen der Strahlungswärmestromdichte oder Intensität. Beginnend mit dem Substrat werden die Systemkoeffizienten durch finite Elemente ermittelt. Wenn die Darstellung zur Oberseite der Anordnung gelangt, wird der Feldamplitude ein auf eins normierter Wert zugewiesen und die übrigen Amplituden schrittweise zurückgerechnet. Das Poynting-Theorem erlaubt es, die normierten Wärmeströme zu berechnen. Nachfolgende Differentiation liefert das Absorptionsvermögen jeder Schicht. Absorbierende Medien, Polarisation, Interferenz und schräger Einfall lassen sich voll berücksichtigen. Die Werte des Absorptionsvermögens pro Schicht braucht man bei der Bestimmung der Kühlungsanforderungen von Laserspiegeln und Absorbern von hochkonzentrierenden Solarkollektoren.

МЕТОД КОНЕЧНЫХ ЭЛЕМЕНТОВ ДЛЯ РАСЧЕТА ОПТИЧЕСКОЙ ИНТЕРФЕРЕНЦИИ

Аннотация — Предложен удобный алгоритм для расчета суммарной отражательной способности многослойной тонкопленочной системы и поглощательной способности отдельных слоев. Для соответствующего учета интерференции задача лучистого переноса формулируется для амплитуды вектора электрического поля, а не для лучистого потока или его интенсивности. Начиная с подложки, коэффициенты системы строятся методом конечных элементов. По достижении верхнего слоя амплитуде поля придается нормированное значение, равное единице, а расчет всех остальных амплитуд проводится методом последовательных приближений. По теореме Пойнтинга можно определить нормированные потоки, а затем дифференцированием определить поглощательную способность каждого слоя. В расчетах учтены поглощение среды, поляризация, интерференция и наклонное падение излучения. Данные об удельной поглощательной способности на один слой необходимы в задачах охлаждения лазерных зеркал и абсорберов, используемых в мощных солнечных коллекторах.